3.4 The Chain Rule

In this section you will learn how to differentiate more difficult functions using the Chain Rule. For example, the function $\boldsymbol{F(x) = \sqrt[3]{x^2+x+1}}$. With the rules we have so far, we cannot differentiate this function.

Notice that the function **F(x)** is a composite function. If we let $y = f(u) = \sqrt[3]{u}$ where $u = g(x) = x^2+x+1$, Then we can write $y = F(x) = f(g(x))$. But we still don't have a rule for how to differentiate composite functions. For this function we need the following new rule.

The Chain Rule: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiatlbe at x and F' is given by the product:

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if y = f(u) and u = g(x) are both differentiable functions, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Example: Find $F'(x)$ if $F(x) = \sqrt[3]{x^2+x+1}$.

Notice that $F(x) = (f \circ g)(x) = f(g(x))$ where $f(u) = \sqrt[3]{u}$ and $g(x) = x^2+x+1$

Since $f'(u) = \frac{1}{3}u^{1-\frac{1}{3}} = \frac{1}{3}u^{\frac{2}{3}} = \frac{\sqrt[3]{u^2}}{3}$ and $g'(x) = 2x+1$,

we get $F'(x) = F'(g(x)) \cdot g'(x) = \sqrt[3]{(x^2+x+1)^2} \cdot (2x+1)$

(An easier way to visualize this is to think of taking the derivative of the "outer" function times the derivative of the "inner" function.)

Example: Differentiate $y = cos\left(\frac{\pi}{x}\right)$. The outer function is the cosine function and the inner function is $\frac{\pi}{x}$.

$$y' = -sin\left(\frac{\pi}{x}\right) \cdot \frac{dy}{dx}\left(\frac{\pi}{x}\right) \qquad \frac{dy}{dx}\left[\frac{\pi}{x}\right] = \frac{x(0)-\pi(1)}{x^2} = -\frac{\pi}{x^2}$$

$$y' = -\sin\left(\frac{\pi}{x}\right)\left(-\frac{\pi}{x^2}\right)$$

$$y' = \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)$$

The Power Rule combined with the Chain Rule: If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{dy}{dx}[u^n] = n \cdot u^{n-1} \cdot u' \qquad \text{or} \qquad \frac{dy}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example: Differentiate

a) $y = (x^2 + 2x)^5$ b) $y = \left(\frac{\cos(x)}{1-2x}\right)^3$ c) $y = (x^2 + 2x)^5 \cdot (1 - 2x)^4$ d) $y = e^{\sec(3x)}$

e) $y = \sin(\sec(\tan(x)))$

Solutions:

a) $y' = 5(x^2 + 2x)^4(2x + 2) = \mathbf{(10x + 10)(x^2 + 2x)^4}$

b) $y' = 3\left(\frac{\cos(x)}{1-2x}\right)^2 \cdot \left(\frac{(1-2x)(-\sin(x)) - (\cos(x)(-2))}{(1-2x)^2}\right) = \frac{\mathbf{3(cos^2(x))[-\sin(x)+2x\sin(x)+2\cos(x)]}}{\mathbf{(1-2x)^4}}$

c) $y' = (x^2 + 2x)^5(4(1 - 2x)^3(-2)) + (1 - 2x)^4(5(x^2 + 2x)^4(2x + 2)$

$y' = \mathbf{-8(x^2 + 2x)^5(1 - 2x)^3 + (10x + 10)(1 - 2x)^4(x^2 + 2x)^4}$

d) $y' = e^{\sec(3x)} \cdot \sec(3x)\tan(3x) \cdot 3$ $\mathbf{y' = 3\sec(3x)\tan(3x)e^{\sec(3x)}}$

e) $y' = \mathbf{\cos(\sec(\tan(x)) \cdot \sec(\tan(x))\tan(\tan(x))\,(sec^2(x)}$

PRACTICE MANY MORE OF THESE PROBLEMS!